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On characteristic twists of multiple Dirichlet series associated to Siegel cusp forms

Özlem Imamoḡlu · Yves Martin

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Abstract We define a twisted two complex variables Rankin-Selberg convolution of Siegel cusp forms of degree 2. We find its group of functional equations and prove its analytic continuation to $\mathbb{C}^2$. As an application we obtain a non-vanishing result for special values of the Fourier Jacobi coefficients. We also prove the analytic properties for the characteristic twists of convolutions of Jacobi cusp forms.

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1 Introduction

In this article we study characteristic twists of the two complex variables Rankin-Selberg convolution of Siegel cusp forms introduced in [7]. We also investigate the analytic properties of a twisted Dirichlet series attached to a pair of Jacobi cusp forms. As an application of our main theorem we obtain a non-vanishing result for the Dirichlet series attached to Fourier Jacobi coefficients of our Siegel forms.

In the case of one complex variable the analytic properties of characteristic twists of Rankin Selberg type Dirichlet series for Siegel modular forms together with their applications to non vanishing results have been studied in a series of papers (see [2,11,12]).

In the study of characteristic twists of several complex variables Dirichlet series, we would like to point out that, other than the expected technical difficulties, a new complication arises.

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Ö. Imamoḡlu (✉)
Department of Mathematics, ETH Zurich, 8096 Zürich, Switzerland
e-mail: ozlem@math.ethz.ch

Y. Martin
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile,
Casilla 653, Santiago, Chile
e-mail: ymartin@uchile.cl

Unlike the papers cited above, in the case of several complex variables, it is *essential* to twist with more than one Dirichlet character as it is done in [4] for other series. It is only then that one can obtain the full group of functional equations and hence the desired analytic continuation to $\mathbb{C}^2$ of the new series.

In order to state precisely our main result, let $F(Z)$ and $G(Z)$ be two Siegel cusp forms of weight k over $Sp_2(\mathbb{Z})$. Each has a Fourier series representation with complex coefficients, say $c(T)$ and $d(T)$, respectively, where $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ runs over the set $\mathcal{J}$ of half-integral, positive definite matrices in $\mathbb{Q}^{2,2}$. Define the multiple Dirichlet series

$$L(F_{\chi,\psi}, G; s, w) = \sum_{\substack{T \in \mathcal{J} \\ r \pmod{2mN}}} \chi(n)\psi(m)c(T)\overline{d(T)}m^{-s}(4\det T)^{-w}, \tag{1}$$

where χ (resp. ψ) is a Dirichlet character mod N (resp. M). This series is absolute and locally uniform convergent on the region $Re(s) > 2$, $Re(w) > k+1$. Define next

$$\Lambda\left(F_{\chi,\psi}, G; s, w\right) = \left(\frac{2\pi}{MN}\right)^{-s} \Gamma(s)L(\overline{\chi}^2\psi^2, 2s)L\left(F_{\chi,\psi}, G; s, w\right) \tag{2}$$

and

$$\begin{aligned} \widetilde{\Lambda}\left(F_{\chi,\psi}, G; s, w\right) = N^{-k+3/2}\left(\frac{\pi}{N}\right)^{-2w+k-3/2} \Gamma(w)\Gamma\left(w-k+\frac{3}{2}\right) \\ \times L(\chi^2, 2w-2k+3)L\left(F_{\chi,\psi}, G; s-w+\frac{1}{2}, w\right), \end{aligned} \tag{3}$$

where $\Gamma(s)$ is Euler's gamma function and $L(\phi, s)$ is the L-function of the Dirichlet character ϕ. Our main theorem gives the analytic properties of (1).

Theorem 1 *Let $F(Z)$ and $G(Z)$ be Siegel cusp forms of weight $k > 1$ over $Sp_2(\mathbb{Z})$. Let M and N be relatively prime positive integers. Let χ (resp. ψ) be a Dirichlet character modulo N (resp M) such that χ, χ^2, $\overline{\chi}^2\psi^2$ are primitive and non-principal. Then the series $L\left(F_{\chi,\psi}, G; s, w\right)$*

i) *admits a holomorphic continuation to $\mathbb{C}^2$, and*
ii) *satisfies the functional equations*

$$\Lambda\left(F_{\chi,\psi}, G; s, w\right) = (-1)^k \frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{M} \Lambda\left(F_{\psi,\chi}, G; 1-s, s+w-\frac{1}{2}\right),$$

$$\widetilde{\Lambda}\left(F_{\chi,\psi}, G; s, w\right) = \left(\frac{\mathcal{G}_\chi}{\sqrt{N}}\right)^4 \widetilde{\Lambda}\left(F_{\overline{\chi},\psi}, G; s, 2k-w-2\right),$$

where $\mathcal{G}_\phi$ denotes the Gauss sum of the Dirichlet character ϕ.

As an application of our main theorem we obtain.

Corollary 1 *Let $F(Z)$ and $G(Z)$ be Siegel cusp forms of weight k for $Sp_2(\mathbb{Z})$ and denote by $f_m(\tau_1, z)$ resp. $g_m(\tau_1, z)$ their Fourier–Jacobi coefficients. Fix any $w \in \mathbb{C}$ with $Re(w) > k+1$, an odd squarefree integer $L \geq 5$ and $\epsilon \in \{\pm 1\}$.*

If there exists m_0 such that $\gcd(m_0, L) = 1$ *and $L(f_{m_0}, g_{m_0}; w) \neq 0$ then there are infinitely many integers m with $L(f_m, g_m; w) \neq 0$ and $\left(\frac{m}{L}\right) = \epsilon$.*

This article is organized as follows: In Sect. 2 we recall the definition and properties of two Eisenstein series that are variations of the Epstein zeta function. They are key ingredients in the integral representation of our Dirichlet series. In Sects. 3 and 5 we define the two twisted Dirichlet series that we study in this paper. One for Jacobi forms and another for Siegel cusp forms (1). In Sects. 4 and 6 we establish the analytic properties of those series. The proof of our main result is in the latter. Finally, in Sect. 7 we indicate how to prove Corollary 1.

Notation If $w \in \mathbb{C}$ then $e(w) = e^{2\pi i w}$. The complex upper half plane is denoted by $\mathcal{H}$. Throughout this article M and N denote positive integers. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\Gamma_0(N)$ or $\Gamma^0(N)$ and χ is a Dirichlet character mod N then $\chi(\gamma) := \chi(d)$. In case that the character χ is primitive, we denote by $\mathcal{G}_\chi$ the Gauss sum associated to it. Throughout, we say that χ is the principal character if it is trivial and primitive. The symbol $\zeta(s)$ is always the Riemann zeta function. If A is any matrix, we denote by ${}^tA$ its transpose and by $\det A$ its determinant. Unless we say otherwise, the entries of any matrix X in $\mathbb{R}^{2,2}$ are labeled as $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. Whenever we write a 4 by 4 matrix as $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the blocks A, B, C, D are in $\mathbb{R}^{2,2}$.

2 Eisenstein series

Let us start with the set $\mathcal{P}$ of symmetric, positive-definite matrices Y in $\mathbb{R}^{2,2}$. Any γ in $GL_2(\mathbb{R})$ acts on $\mathcal{P}$ via $Y[\gamma] = {}^t\gamma Y \gamma$. Notice that $\mathcal{H}$ and $\mathcal{SP} = \{Y \in \mathcal{P} \mid \det Y = 1\}$ can be identified as $SL_2(\mathbb{R})$-spaces by the map

$$\tau = x + iy \to P_\tau = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right], \quad \text{where } x = Re(\tau),\ y = Im(\tau). \tag{4}$$

Conversely, if $Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_4 \end{pmatrix}$ is in $\mathcal{P}$ and $\det Y = t^2$, then $t^{-1}Y \in \mathcal{SP}$. The inverse image of the latter under the mapping (4) is $\tau_Y = y_4^{-1}(y_2 + it)$.

Let (u_1, u_2) and (v_1, v_2) be in $\mathbb{Q}^2$. For any Y in $\mathcal{P}$ and s in $\mathbb{C}$ let

$$\zeta_{(u_1,u_2),\,(v_1,v_2)}(Y; s) = \sum_{\substack{(l,c)\in\mathbb{Z}^2 \\ (l+v_1,c+v_2)\neq(0,0)}} e(u_1 l + u_2 c) \left(Y \begin{bmatrix} l + v_1 \\ c + v_2 \end{bmatrix} \right)^{-s}. \tag{5}$$

This series is the Epstein zeta function. It is absolute and locally uniform convergent on the half-plane $Re(s) > 1$. The product $\pi^{-s}\Gamma(s)\zeta_{(u_1,u_2),\,(v_1,v_2)}(Y; s)$ has an analytic continuation to $\mathbb{C}$. This is entire if $(u_1, u_2) \notin \mathbb{Z}^2$. Otherwise it is holomorphic on $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$. The residue at this point is $(\det Y)^{-1/2}$. Moreover

$$\begin{aligned} &e(u_1 v_1 + u_2 v_2)\pi^{-s}\Gamma(s)\zeta_{(u_1,u_2),\,(v_1,v_2)}(Y; s) \\ &\quad = (\det Y)^{-1/2}\pi^{-(1-s)}\Gamma(1-s)\zeta_{(v_1,v_2),\,(-u_1,-u_2)}\left(Y^{-1}; 1-s\right). \end{aligned} \tag{6}$$

For more details about (5), see, for example [15, pp. 60–71] or [12, p. 493]. The first Eisenstein series that we recall involves the groups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0\ (N) \right\} \text{ and } \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \mid l \in \mathbb{Z} \right\}.$$

Definition 1 Let N and N' be positive integers with $N|N'$, ϕ a Dirichlet character modulo N and s a complex variable. For any τ in $\mathcal{H}$ set

$$E_{N',\phi}(\tau,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N')}\phi(\gamma)\,(Im\gamma(\tau))^s\,. \tag{7}$$

This series is absolute and locally uniform convergent on the region $Re(s)>1$. Moreover, $E_{N',\phi}(\gamma(\tau),s)=\overline{\phi}(\gamma)E_{N',\phi}(\tau,s)$ for any $\gamma\in\Gamma_0(N')$.

Lemma 1 *If $Re(s)>1$ then*

$$2L(\phi,2s)E_{N',\phi}(\tau,s)=N'^{-2s}\sum_{r=1}^{N'}\phi(r)\zeta_{(0,0),(0,r/N')}(P_\tau;s), \tag{8}$$

where $\tau\in\mathcal{H}$ and $P_\tau\in\mathcal{P}$ are related as in (4).

If ϕ is a primitive, non-principal character then $\pi^{-s}\Gamma(s)L(\phi,2s)E_{N',\phi}(\tau,s)$ admits a holomorphic continuation to $\mathbb{C}$. If ϕ is the principal Dirichlet character and $N'=N=1$ then $2L(\phi,2s)E_{N',\phi}(\tau,s)=2\zeta(2s)E(\tau,s)$ where $E(\tau,s)$ is the weight zero Eisenstein series associated to $SL_2(\mathbb{Z})$.

Proof For any $\tau=x+iy$ in $\mathcal{H}$,

$$\begin{aligned}2L(\phi,2s)E_{N',\phi}(\tau,s)&=\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}}\phi(d)\frac{y^s}{|cN'\tau+d|^{2s}}\\&=\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}}\phi(d)\left(y(cN')^2+\frac{(cN'x+d)^2}{y}\right)^{-s}=\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}}\phi(d)\left(P_\tau\begin{bmatrix}cN'\\d\end{bmatrix}\right)^{-s}.\end{aligned}$$

If we write every d above as $d=r+lN'$ with $r\pmod{N'}$, $l\in\mathbb{Z}$, we get (5).

The second part of the lemma for ϕ primitive and non-principal follows from the equation above, the analytic continuation of the Epstein zeta function and the orthogonality relation of Dirichlet characters. If ϕ is the principal character the equation is clear and the statement well-known. □

Let χ be a Dirichlet character mod N. In this work we use the series $E_{N^2,\chi^2}(\tau,s)$ and $E_{N,\chi^2}(\tau,s)$. They are related, as we can see in the next proposition if we set

$$\mathcal{E}_{N',\chi^2}(\tau,s)=\left(\frac{N^2}{\pi}\right)^s\Gamma(s)L(\chi^2,2s)E_{N',\chi^2}(\tau,s)$$

for $N'=N,N^2$ and $W_{N^2}=\begin{pmatrix}0&-1/N\\N&0\end{pmatrix}$.

Proposition 1 *Let χ be a primitive, non-principal Dirichlet character mod N such that χ^2 is also primitive and non-principal mod N. Then $\mathcal{E}_{N,\chi^2}(\tau,s)$ and $\mathcal{E}_{N^2,\chi^2}(\tau,s)$ are entire functions of s, they satisfy*

$$\mathcal{E}_{N^2,\chi^2}(\tau,s)=\frac{\mathcal{G}_{\chi^2}}{N}\mathcal{E}_{N,\overline{\chi^2}}(W_{N^2}\tau,1-s)$$

and $\mathcal{E}_{N^2,\chi^2}(\tau,s)=O(y^{Re(s)})$ as $y=Im(\tau)\to\infty$.

If χ_0 *is the principal character then* $\mathcal{E}_{1,\chi_0}(\tau, s) = \pi^{-s}\Gamma(s)\zeta(2s)E_{1,\chi_0}(\tau, s)$ *is a series with a holomorphic continuation to* $\mathbb{C} - \{0, 1\}$. *At* $s = 0, 1$ *it has simple poles with residue* $Res_{s=1}\mathcal{E}_{1,\chi_0}(\tau, s) = 1$. *Moreover,* $\mathcal{E}_{1,\chi_0}(\tau, s) = \mathcal{E}_{1,\chi_0}(\tau, 1-s)$ *and* $\mathcal{E}_{1,\chi_0}(\tau, s) = O(y^\sigma)$ *as* $y \to \infty$ *where* $\sigma = max\{Re(s), 1 - Re(s)\}$.

Proof The first part of the proposition follows from Lemma 1. Also, from (8) and the functional equation (6) of the Epstein zeta function one gets

$$\begin{aligned} &2\pi^{-s}\Gamma(s)L(\chi^2, 2s)E_{N^2,\chi^2}(\tau, s) \\ &\quad = N^{-4s}\pi^{-(1-s)}\Gamma(1-s)\sum_{r=0}^{N^2-1}\chi^2(r)\zeta_{(0,r/N^2),(0,0)}(P_\tau^{-1}; 1-s). \end{aligned} \tag{9}$$

Using (5) and

$$\sum_{r=0}^{N^2-1}\chi^2(r)e\left(\frac{dr}{N^2}\right) = \begin{cases} N\mathcal{G}_{\chi^2}\bar{\chi}^2(d/N) & \text{if } N|d, \\ 0 & \text{otherwise} \end{cases}$$

which holds as χ^2 is primitive, one obtains

$$\sum_{r=0}^{N^2-1}\chi^2(r)\zeta_{(0,r/N^2),(0,0)}(P_\tau^{-1}; s) = N\mathcal{G}_{\chi^2}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}}\bar{\chi}^2(d)\left(P_\tau^{-1}\begin{bmatrix} c \\ dN \end{bmatrix}\right)^{-s}$$

whenever $Re(s) > 1$. On the other hand $P_{\gamma(\tau)} = P_\tau[{}^t\gamma]$ for all $\gamma \in SL_2(\mathbb{R})$. Therefore

$$P_\tau^{-1}\begin{bmatrix} c \\ dN \end{bmatrix} = P_{W_{N^2}(\tau)}\begin{bmatrix} cN \\ d \end{bmatrix}.$$

From these equations and Lemma 1 we deduce

$$\sum_{r=0}^{N^2-1}\chi^2(r)\zeta_{(0,r/N^2),(0,0)}(P_\tau^{-1}; s) = 2N\mathcal{G}_{\chi^2}L(\chi^2, 2s)E_{N,\bar{\chi}}(W_{N^2}(\tau), s). \tag{10}$$

Now (8), (9), (10) and the analytic continuation of $\mathcal{E}_{N,\chi^2}(\tau, s)$ yield the functional equation in the proposition. As for the asymptotic behavior of $\mathcal{E}_{N^2,\chi^2}(\tau, s)$, notice that

$$2L(\chi^2, s)E_{N^2,\chi^2}(\tau, s) = y^s\sum_{\substack{c,d\ \mathbb{Z} \\ (c,d)\neq(0,0)}}\chi^2(d)|c(N^2\tau) + d|^{-2s}.$$

The latter is a classical Eisenstein series whose Fourier expansion in terms of powers and Bessel functions is well-known (see, for example [14]). From it one gets the desired asymptotic. The statements about $\mathcal{E}_{1,\chi_0}(\tau, z)$ are those of a classical Eisenstein series (see [3]). □

Next we define a second kind of Eisenstein series. Start with the complex-valued function

$$p_{s,w}(Y) = y_1^s(\det Y)^w, \quad \text{where} \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_4 \end{pmatrix} \in \mathcal{P} \tag{11}$$

and s, w are in $\mathbb{C}$ (for details on this map see [16, vol. II]). Then recall the notation

$$\Gamma^0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, b \equiv 0\ (N)\right\}, \quad \Gamma_0^0(M, N) = \Gamma_0(M) \cap \Gamma^0(N),$$

$$\Gamma_\infty^0(N) = \Gamma_\infty \cap \Gamma^0(N) \quad \text{and} \quad \Gamma_0^\infty(N) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Gamma_\infty^0(N)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2 Let ϕ be a Dirichlet character modulo MN such that $\phi(-1) = 1$. Let s, w be two complex variables. For any Y in $\mathcal{P}$ set

$$E_\phi(Y; s, w) = \sum_{\gamma \in \Gamma_0^0(M,N)/\Gamma_\infty^0(N)} \overline{\phi}(\gamma) p_{-s,-w}(Y[\gamma]). \tag{12}$$

This series has been extensively studied if ϕ is the principal character [13,16]. In particular, the comparison with the latter implies that (12) is absolute and locally uniform convergent on the region $Re(s) > 1$. Moreover,

$$E_\phi(Y[\gamma]; s, w) = \phi(\gamma) E_\phi(Y; s, w) \quad \text{for} \quad \gamma \in \Gamma_0^0(M, N).$$

Let $\mathcal{H}_2 = \left\{Z \in \mathbb{C}^{2,2} \mid {}^tZ = Z,\ ImZ \,\text{pos. def}\right\}$. The symplectic group $Sp_2(\mathbb{R})$ acts on $\mathcal{H}_2$ via $\gamma Z = (AZ + B)(CZ + D)^{-1}$, where we write $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Definition 3 Let $\mathcal{C}(N, M)$ be the group of matrices

$$\mathcal{C}(N, M) = \left\{ \begin{pmatrix} {}^tA & {}^tAB \\ 0 & A^{-1} \end{pmatrix} \in Sp_2(\mathbb{Z}) \mid A \in \Gamma_0^0(M, N) \right\}.$$

The action of $Sp_2(\mathbb{R})$ on $\mathcal{H}_2$ induces an action of $\mathcal{C}(N, M)$ on $\mathcal{H}_2$ and the action of $GL_2(\mathbb{R})$ on $\mathcal{P}$ induces an action of $\Gamma_0^0(M, N)$ on $\mathcal{P}$. The correspondence $Z \to Y = ImZ$ is a well-defined map that preserve those actions. Furthermore,

$$E_\phi\left(Im(\gamma Z); s, w\right) = \phi(A) E_\phi\left(Im(Z); s, w\right)$$

for any $\gamma = \begin{pmatrix} {}^tA & {}^tAB \\ 0 & A^{-1} \end{pmatrix} \in \mathcal{C}(N, M)$ and (s, w) in the domain of convergence.

Lemma 2 *If $Re(s) > 1$ then*

$$2L(\phi, 2s) E_\phi(Y; s, w) = M^{-2s} (\det Y)^{-w} \sum_{r=1}^{MN} \phi(r) \zeta_{(0,0),\,(\frac{r}{MN},0)} \left(Y \left[\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \right]; s \right). \tag{13}$$

If ϕ is a primitive, non-principal character then $\pi^{-s}\Gamma(s)L(\phi, 2s)E_\phi(Y; s, w)$ admits a holomorphic continuation to $\mathbb{C}^2$.

If ϕ is the principal character then $MN = 1$ and

$$L(\phi, 2s) E_\phi(Y; s, w) = (\det Y)^{-w-s/2} \zeta(2s) E(\tau_Y, s)$$

where $E(\tau, s)$ is the classical weight zero Eisenstein series associated to $SL_2(\mathbb{Z})$ and τ_Y is the element of $\mathcal{H}$ associated to Y in remark below (4).

Proof The elements of $\Gamma_0^0(M, N)/\Gamma_\infty^0(N)$ are in one to one correspondence with the tuples $(a, c) \in \mathbb{Z}^2/\pm 1$ with $M|c$ and $\gcd(a, Nc) = 1$. Hence

$$2E_\phi(Y; s, w) = (\det Y)^{-w} \sum_{\substack{(a,e)\in\mathbb{Z}^2 \\ \gcd(a, MNe)=1}} \phi(a) \left(Y \begin{bmatrix} a \\ Me \end{bmatrix} \right)^{-s}. \tag{14}$$

Therefore

$$2L(\phi, 2s)E_{\phi}(Y; s, w) = (\det Y)^{-w} \sum_{\substack{(a,e)\in\mathbb{Z}^2 \\ (a,e)\neq(0,0)}} \phi(a) \left(Y\left[\begin{matrix} a \\ Me \end{matrix}\right]\right)^{-s}$$

$$= M^{-2s} (\det Y)^{-w} \sum_{r=1}^{MN} \phi(r) \sum_{\substack{(l,e)\in\mathbb{Z}^2 \\ (lM+\frac{r}{N},e)\neq(0,0)}} \left(Y\left[\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{r}{MN}+l \\ e \end{pmatrix}\right]\right)^{-s},$$

and (13) follows. As in the proof of the first lemma, the statement for ϕ primitive and non-principal follows from Eq. (13), the analytic continuation of the Epstein zeta function and the orthogonality relation of Dirichlet characters.

On the other hand, if ϕ is the principal character then $M = N = 1$ and

$$2L(\phi, 2s)E_{\phi}(Y; s, w) = (\det Y)^{-w} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \left(Y\left[\begin{pmatrix} c \\ d \end{pmatrix}\right]\right)^{-s}.$$

If we write Y as in (11) and put $\det Y = t^2$, the previous equation is

$$2L(\phi, 2s)E_{\phi}(Y; s, w) = t^{-2w-s} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \left(\frac{t}{y_4}\right)^{s} \left|c\left(\frac{y_2}{y_4} + i\frac{t}{y_4}\right) + d\right|^{-2s}.$$

□

The arguments in this article use the Eisenstein series $E_{\overline{\chi}^2\psi^2}(Y; s, w)$ where χ (resp. ψ) is a Dirichlet character mod N (resp. mod M). For simplicity we assume the condition $\gcd(M, N) = 1$ from now on and set

$$\mathcal{E}_{\overline{\chi}^2\psi^2}(Y; s, w) = M^{s}(MN)^{-w}\pi^{-s}\Gamma(s)L(\overline{\chi}^2\psi^2; 2s)E_{\overline{\chi}^2\psi^2}(Y; s, w). \tag{15}$$

Proposition 2 *Let χ and ψ be Dirichlet characters as above such that $\overline{\chi}^2\psi^2$ is primitive and non-principal. Then $\mathcal{E}_{\overline{\chi}^2\psi^2}(Y; s, w)$ admits a holomorphic continuation to $\mathbb{C}^2$ and satisfies*

$$\mathcal{E}_{\overline{\chi}^2\psi^2}(Y; s, w) = \frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}}\mathcal{E}_{\chi^2\overline{\psi}^2}\left(Y^{-1}\left[\begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{M} \end{pmatrix}\right]; 1-s, -w-\frac{1}{2}\right).$$

Proof Since $\overline{\chi}^2\psi^2$ is primitive and non-principal, the holomorphic continuation of $\mathcal{E}_{\overline{\chi}^2\psi^2}(Y; s, w)$ to $\mathbb{C}^2$ is a consequence of Lemma 2. For the proof of the identity, observe that (13) and the functional equation of the Epstein zeta function yield

$$2\pi^{-s}\Gamma(s)L(\overline{\chi}^2\psi^2, 2s)E_{\overline{\chi}^2\psi^2}(Y; s, w) = \pi^{-(1-s)}\Gamma(1-s)M^{-2s}N^{-1}$$

$$\times (\det Y)^{-w-\frac{1}{2}} \sum_{r=1}^{MN} \overline{\chi}^2\psi^2(r)\zeta_{(\frac{r}{MN},0),\,(0,0)}\left(Y^{-1}\left[\begin{pmatrix} \frac{1}{N} & 0 \\ 0 & 1 \end{pmatrix}\right]; 1-s\right). \tag{16}$$

On the other hand, for arbitrary s with $Re(s) >> 0$ and primitive $\overline{\chi}^2\psi^2$

$$\sum_{r=1}^{MN} \overline{\chi}^2\psi^2(r)\zeta_{(\frac{r}{MN},0),\,(0,0)}(Y;s)$$

$$= \sum_{\substack{(l,e)\in\mathbb{Z}^2\\(l,e)\neq(0,0)}} \left(\sum_{r=1}^{MN}\overline{\chi}^2\psi^2(r)e\left(\frac{rl}{MN}\right)\right)\left(Y\begin{bmatrix}l\\e\end{bmatrix}\right)^{-s}$$

$$= \mathcal{G}_{\overline{\chi}^2\psi^2} L(\chi^2\overline{\psi}^2, 2s) \sum_{\substack{(l,e)\in\mathbb{Z}^2\\ \gcd(l,MNe)=1}} \chi^2\overline{\psi}^2(l)\left(Y\left[\begin{pmatrix}1&0\\0&\frac{1}{M}\end{pmatrix}\begin{pmatrix}l\\Me\end{pmatrix}\right]\right)^{-s}$$

$$= 2\mathcal{G}_{\overline{\chi}^2\psi^2}\left(\det Y\left[\begin{pmatrix}1&0\\0&\frac{1}{M}\end{pmatrix}\right]\right)^{w} L(\chi^2\overline{\psi}^2, 2s)E_{\chi^2\overline{\psi}^2}\left(Y\left[\begin{pmatrix}1&0\\0&\frac{1}{M}\end{pmatrix}\right];s,w\right).$$

(This last step is a direct consequence of (14)). Thus

$$\sum_{r=1}^{MN}\overline{\chi}^2\psi^2(r)\zeta_{(\frac{r}{MN},0),\,(0,0)}\left(Y^{-1}\left[\begin{pmatrix}\frac{1}{N}&0\\0&1\end{pmatrix}\right];1-s\right) = 2\mathcal{G}_{\overline{\chi}^2\psi^2}(MN)^{2w+1}$$
$$\times(\det Y)^{w+\frac{1}{2}}L(\chi^2\overline{\psi}^2, 2(1-s))E_{\chi^2\overline{\psi}^2}\left(Y^{-1}\left[\begin{pmatrix}\frac{1}{N}&0\\0&\frac{1}{M}\end{pmatrix}\right];1-s,-w-\frac{1}{2}\right). \quad (17)$$

Now, we just put together (16) and (17) to get

$$\pi^{-s}\Gamma(s)L(\overline{\chi}^2\psi^2, 2s)E_{\overline{\chi}^2\psi^2}(Y;s,w) = M^{-2s}N^{-1}\pi^{-(1-s)}\Gamma(1-s)$$
$$\times\mathcal{G}_{\overline{\chi}^2\psi^2}L(\chi^2\overline{\psi}^2, 2(1-s))(MN)^{2w+1}E_{\chi^2\overline{\psi}^2}\left(Y^{-1}\left[\begin{pmatrix}\frac{1}{N}&0\\0&\frac{1}{M}\end{pmatrix}\right];1-s,-w-\frac{1}{2}\right).$$

The desired functional equation is a rearrangement of this expression. □

3 A twisted convolution of Jacobi cusp forms

For any δ in $SL_2(\mathbb{Z})$, (λ,μ) in $\mathbb{Z}^2$, $(\tau,z)\in\mathcal{H}\times\mathbb{C}$ and positive integers k, m, define

$$J([\delta,\lambda,\mu];\tau,z) = (c\tau+d)^{-k}e^{m}\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right). \quad (18)$$

This is a 1-cocycle. The group $\Gamma^J = SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$ acts on the set of functions $f(\tau,z):\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ via

$$f|_{k,m}[\delta,\lambda,\mu](\tau,z) = J([\delta,\lambda,\mu];\tau,z)f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right). \quad (19)$$

This can be extended to an action of $SL_2(\mathbb{R})\ltimes(\mathbb{R}^2\cdot S^1)$, where S^1 is the unit circle in $\mathbb{C}$. A Jacobi cusp form of weight k and index m over Γ^J is a holomorphic function $f(\tau,z):\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ such that $f|_{k,m}[\delta,\lambda,\mu](\tau,z) = f(\tau,z)$ for all $[\delta,\lambda,\mu]\in\Gamma^J$ and equal to a Fourier series

$$f(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\4mn>r^2}} c(n,r)e(n\tau)e(rz). \quad (20)$$

Jacobi cusp forms of weight k and index m over subgroups of Γ^J are defined similarly (see for example [5, p. 9]). Let N be a positive integer and χ any Dirichlet character modulo N. If $f(\tau, z)$ is a Jacobi cusp form as above, define

$$f_\chi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>r^2}} \chi(n)c(n,r)e(n\tau)e(rz). \tag{21}$$

It is easy to check that for primitive χ one has

$$\mathcal{G}_{\overline{\chi}} f_\chi(\tau, z) = \sum_{\mu\,(N)} \overline{\chi}(\mu) f\,|_{k,m}\left[\begin{pmatrix}1 & \mu/N\\ 0 & 1\end{pmatrix}, 0, 0\right](\tau, z), \tag{22}$$

From this identity follows that $f_\chi(\tau, z)$ is a Jacobi cusp form of weight k, index m and character χ^2 over the group $\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z})$.

Define next $c_r(D, \chi) = \chi(n)c(n, r)$ whenever $D = 4mn - r^2$. The invariance of $f_\chi(\tau, z)$ under $[Id, N\lambda, 0]$ for any λ in $\mathbb{Z}$ yields $c_r(D, \chi) = c_{r'}(D, \chi)$ whenever $r \equiv r' \pmod{2mN}$. This identity together with simple manipulations of (21) imply

$$f_\chi(\tau, z) = \sum_\mu f_{\chi,\mu}(\tau)\Theta_{mN,\mu}(N\tau, z), \tag{23}$$

where the sum is over the integers μ mod $2mN$ and

$$f_{\chi,\mu}(\tau) = \sum_{D=1}^{\infty} c_\mu(D, \chi)e\left(\frac{D}{4m}\tau\right), \tag{24}$$

$$\Theta_{mN,\mu}(\tau, z) = \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu\,(2mN)}} e\left(\frac{r^2}{4mN}\tau + rz\right). \tag{25}$$

Let $g(\tau, z)$ be second Jacobi cusp form of weight k and index m over the group Γ^J, say $g(\tau, z) = \sum_{n,r} d(n,r)e(n\tau)e(rz)$. If we put $d_r(D) = d(n, r)$ whenever $D = 4mn - r^2$ we have

$$g(\tau, z) = \sum_{\nu\,(2m)} g_\nu(\tau)\Theta_{m,\nu}(\tau, z) \quad \text{with} \quad g_\nu(\tau) = \sum_{D=1}^{\infty} d_\nu(D)e\left(\frac{D}{4m}\tau\right). \tag{26}$$

Obviously, we can also write (26) as $g(\tau, z) = \sum_{\nu\,(2mN)} g_\nu(\tau)\Theta_{mN,\nu}(N\tau, z)$.

Definition 4 For Jacobi cusp forms $f(\tau, z)$ and $g(\tau, z)$ as above and a Dirichlet character χ modulo N, define

$$L(f_\chi, g; s) = \sum_{D=1}^{\infty} \sum_{\mu\,(2mN)} c_\mu(D, \chi)\overline{d_\mu(D)}D^{-s}. \tag{27}$$

Since $f(\tau, z)$ and $g(\tau, z)$ are cusp forms one has $c_\mu(D, \chi) = O(D^{k/2})$ and $d_\mu(D) = O(D^{k/2})$ for all μ, D. Thus $L(f_\chi, g; s)$ is an absolute and locally uniform convergent series on the region $Re(s) > k + 1$. This convolution was studied in [7] in the case where χ is the principal character.

In order to give an integral representation for (27) we identify any $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ with the tuple of real coordinates (x, y, p, q) via the equations $\tau = x + iy$ and $z = p\tau + q$. Then we notice that

$$h_{f_\chi,g}(\tau, z) = y^k e^m(2p^2 iy) f_\chi(\tau, z)\overline{g(\tau, z)} \tag{28}$$

is invariant under the action (19) whenever δ is in $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \mid l \in \mathbb{Z} \right\}$ and (λ, μ) is in $N\mathbb{Z} \times \mathbb{Z}$. The volume element $y^{-2} dq dx dp dy$ of $\mathcal{H} \times \mathbb{C}$ is also Γ^J-invariant, thus the integral in the next lemma is well-defined.

Lemma 3 *For any s in $\mathbb{C}$ with $Re(s) > k + 1$,*

$$\int_{\mathcal{H}\times\mathbb{C}/\Gamma_\infty \ltimes (N\mathbb{Z}\times\mathbb{Z})} h_{f_\chi, g}(\tau, z) y^{s-2} dq dx dp dy$$
$$= \frac{1}{\sqrt{m}} \left(\frac{\pi}{m}\right)^{-s-k+\frac{3}{2}} \Gamma\left(s + k - \frac{3}{2}\right) L\left(f_\chi, g; s + k - \frac{3}{2}\right). \tag{29}$$

Proof We may compute this integral using the $\Gamma_\infty \ltimes (N\mathbb{Z} \times \mathbb{Z})$-fundamental domain

$$\left\{(x, y, p, q) \in \mathbb{R}^4 \mid 0 \le x, q \le 1,\ 0 < y < \infty,\ 0 \le p \le N\right\}. \tag{30}$$

We omit the details since the argument is very similar to the proof of Lemma 2 in [7]. □

4 Analytic properties of $L(f_\chi, g; s)$

The purpose of this section is to prove the meromorphic continuation and the functional equation of $L(f_\chi, g; s)$.

Lemma 4 *If $\delta \in SL_2(\mathbb{Q})$ then*

$$h_{f_\chi, g}(\delta(\tau, z)) = y^k e^m\left(2p^2 iy\right) f_\chi|_{k,m}[\delta, 0, 0](\tau, z)\overline{g|_{k,m}[\delta, 0, 0](\tau, z)}.$$

In particular $h_{f_\chi, g}(\gamma(\tau, z)) = \chi^2(\gamma) h_{f_\chi, g}(\tau, z)$ for any γ in $\Gamma_0(N^2)$.

The proof of this fact is a straightforward computation. It uses that (18) is a cocycle and

$$y^k e^m\left(2p^2 iy\right) = (Im(\tau))^k e^m\left(\frac{(z - \overline{z})^2}{\tau - \overline{\tau}}\right).$$

We omit the details. This lemma together with the properties of the Eisenstein series (7) allow us to express (29) as

$$\int_{\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes (N\mathbb{Z}\times\mathbb{Z})} h_{f_\chi, g}(\tau, z) E_{N^2, \chi^2}(\tau, s) y^{-2} dq dx dp dy$$
$$= \frac{1}{\sqrt{m}} \left(\frac{\pi}{m}\right)^{-s-k+\frac{3}{2}} \Gamma\left(s + k - \frac{3}{2}\right) L\left(f_\chi, g; s + k - \frac{3}{2}\right). \tag{31}$$

Next, for any Dirichlet character χ mod N set

$$A_\chi = \sum_{\substack{\mu, \nu\, (N) \\ \gcd(\mu, N) = \gcd(\nu, N) = 1 \\ \mu\mu^* \equiv \nu\nu^* \equiv 1\, (N)}} \chi(\mu - \nu) e\left((\nu^* - \mu^*)/N\right).$$

Lemma 5 *For any s in $\mathbb{C}$ with $Re(s) >> 0$,*

$$\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})} h_{f_\chi,g}(\tau,z)E_{N,\overline{\chi}^2}\left(W_{N^2}\tau,s\right)y^{-2}dqdxdpdy$$
$$=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\frac{1}{\sqrt{m}}\left(\frac{\pi}{m}\right)^{-s-k+3/2}\Gamma\left(s+k-\frac{3}{2}\right)L\left(f_{\overline{\chi}},g;s+k-\frac{3}{2}\right). \tag{32}$$

Proof The integral in the left hand side of (32) is equal to

$$\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})} h_{f_\chi,g}\left(W_{N^2}^{-1}(\tau,z)\right)E_{N,\overline{\chi}^2}(\tau,s)y^{-2}dqdxdpdy. \tag{33}$$

Now set $\Gamma_0(N)=\cup_{j=1}^{N}\Gamma_0(N^2)M_j$ with $M_j=\begin{pmatrix}1&0\\jN&1\end{pmatrix}$. Any fundamental domain $\mathcal{F}$ for the action of $\Gamma_0(N)\ltimes(N\mathbb{Z}\times\mathbb{Z})$ on $\mathcal{H}\times\mathbb{C}$ defines the fundamental domain $\cup_{j=1}^{N}[M_j,0,0]\mathcal{F}$ of $\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})$ on $\mathcal{H}\times\mathbb{C}$. This fact and the invariance of $E_{N,\overline{\chi}^2}(\tau,s)$ under M_j allow us to write (33) as

$$\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_0(N)\ltimes(N\mathbb{Z}\times\mathbb{Z})}\left(\sum_{j=1}^{N}h_{f_\chi,g}\left(W_{N^2}^{-1}M_j(\tau,z)\right)\right)E_{N,\overline{\chi}^2}(\tau,s)y^{-2}dqdxdpdy.$$

Now, for any δ in $\Gamma_0(N)$ and $1\le j\le N$ there is δ' in $\Gamma_0(N^2)$ and $1\le l\le N$ such that $W_{N^2}^{-1}M_j\delta=\delta'W_{N^2}^{-1}M_l$. Moreover $\chi(\delta')=\overline{\chi}(\delta)$. From these facts, Lemma 4 and the definition of $E_{N,\overline{\chi}^2}(\tau,s)$ we deduce that the previous integral is equal to

$$\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_\infty\ltimes(N\mathbb{Z}\times\mathbb{Z})}\left(\sum_{j=1}^{N}h_{f_\chi,g}\left(W_{N^2}^{-1}M_j(\tau,z)\right)\right)y^{s-2}dqdxdpdy$$
$$=\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_\infty\ltimes(N\mathbb{Z}\times\mathbb{Z})}\left(\sum_{j=1}^{N}h_{f_\chi,g}\left(T_{-j}W_{N^2}^{-1}(\tau,z)\right)\right)y^{s-2}dqdxdpdy. \tag{34}$$

where $T_{-j}=W_{N^2}^{-1}M_jW_{N^2}=\begin{pmatrix}1&-j/N\\0&1\end{pmatrix}$. In the following we use Lemma 4 and the fundamental domain (30) in order to compute this integral. Since χ is primitive, by (22) one has

$$\sum_{j=1}^{N}f_\chi|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)\overline{g|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)}$$
$$=\frac{1}{\mathcal{G}_{\overline{\chi}}}\sum_{\mu\,(N)}\sum_{\nu\,(N)}\overline{\chi}(\mu-\nu)f|_{k,m}[T_\mu W_{N^2}^{-1},0,0](\tau,z)\overline{g|_{k,m}[T_\nu W_{N^2}^{-1},0,0](\tau,z)}.$$

Put $l=\gcd(\mu,N)$ and $t=\gcd(\nu,N)$. Also write $l\mu$ (resp. $t\nu$) in place of μ (resp. ν). Then the last expression is equal to

$$\frac{1}{\mathcal{G}_{\overline{\chi}}}\sum_{\substack{l|N,\,t|N\\ \gcd(l,t)=1}}\ \sum_{\substack{\mu\,(N/l),\,\nu\,(N/t)\\ \gcd(\mu,N/l)=\gcd(\nu,N/t)=1}} \overline{\chi}(l\mu-t\nu)f|_{k,m}[T_{l\mu}W_{N^2}^{-1},0,0](\tau,z) \times\overline{g|_{k,m}[T_{t\nu}W_{N^2}^{-1},0,0](\tau,z)}. \tag{35}$$

Let μ^*, ν^* be integers such that $\mu\mu^*\equiv 1 \pmod{N/l}$ and $\nu\nu^*\equiv 1 \pmod{N/t}$. Then

$$f|_{k,m}[T_{l\mu}W_{N^2}^{-1},0,0](\tau,z)=f|_{k,m}\left[\begin{pmatrix} l & -\mu^*/N\\ 0 & 1/l\end{pmatrix},0,0\right](\tau,z),$$

$$g|_{k,m}[T_{l\mu}W_{N^2}^{-1},0,0](\tau,z)=g|_{k,m}\left[\begin{pmatrix} t & -\nu^*/N\\ 0 & 1/t\end{pmatrix},0,0\right](\tau,z).$$

These remarks show that (35) can be written as

$$\frac{1}{\mathcal{G}_{\overline{\chi}}}\sum_{\substack{l|N,\,t|N\\ \gcd(l,t)=1}}(lt)^k\sum_{\substack{\mu\,(N/l),\,\nu\,(N/t)\\ \gcd(\mu,N/l)=\gcd(\nu,N/t)=1}}\overline{\chi}(l\mu-t\nu)f\left(l^2\tau-\frac{l\mu^*}{N},lz\right)\overline{g\left(t^2\tau-\frac{t\nu^*}{N},tz\right)}.$$

Using this and the Fourier representation of $f(\tau,z)$ and $g(\tau,z)$ we conclude

$$\begin{aligned}&\int_0^1\int_0^1\sum_{j=1}^{N}f_\chi|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)\overline{g|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)}dqdx\\ &=\frac{1}{\mathcal{G}_{\overline{\chi}}}\sum_{\substack{l|N,\,t|N\\ \gcd(l,t)=1}}(lt)^k\sum_{\substack{\mu\,(N/l),\,\nu\,(N/t)\\ \gcd(\mu,N/l)=\gcd(\nu,N/t)=1}}\overline{\chi}(l\mu-t\nu)\sum_{\substack{j,j'\in\mathbb{Z}\\ 4mj'>j^2}}c(t^2j',tj)\\ &\quad\times\overline{d(l^2j',lj)}e\left(((lt)j'+jp)ltiy-\frac{lt^2j'\mu^*}{N}\right)\overline{e\left(((lt)j'+jp)ltiy-\frac{l^2tj'\nu^*}{N}\right)}.\end{aligned}$$

Arguing exactly as in [12, p. 497], the fact that χ is non-principal and primitive yields

$$\begin{aligned}&\int_0^1\int_0^1\sum_{j=1}^{N}f_\chi|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)\overline{g|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)}dqdx\\ &=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\sum_{\substack{j,j'\in\mathbb{Z}\\ 4mj'>j^2}}\overline{\chi}(j')c(j',j)\overline{d(j',j)}e\left(2(j'+jp)iy\right).\end{aligned}$$

Next we compute

$$\begin{aligned}&\int_0^N e^m(2p^2iy)\int_0^1\int_0^1\sum_{j=1}^{N}f_\chi|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)\overline{g|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)}dqdxdp\\ &=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>r^2}}\overline{\chi}(n)c(n,r)\overline{d(n,r)}e(2niy)\int_0^N e\left(2(mp^2+rp)iy\right)dp\\ &=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\sum_{\mu(2mN)}\sum_{D=1}^{\infty}c_\mu(D,\overline{\chi})\overline{d_\mu(D)}e\left(\frac{D}{2m}iy\right)\sum_{l\in\mathbb{Z}}\int_0^N e^{2m}\left(\left(p+lN+\frac{\mu}{2m}\right)^2iy\right)dp.\end{aligned}$$

The inner sum is equal to $(my)^{-1/2}$. Thus the integrals in (34) are equal to

$$\int_0^\infty\int_0^N\int_0^1\int_0^1 y^k e^m(2p^2iy)\sum_{j=1}^N f_\chi|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)$$
$$\times\overline{g|_{k,m}[T_{-j}W_{N^2}^{-1},0,0](\tau,z)}y^{s-2}dqdxdpdy$$
$$=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\sum_{\mu\,(2mN)}\sum_{D=1}^\infty c_\mu(D,\overline{\chi})\overline{d_\mu(D)}\frac{1}{\sqrt{m}}\int_0^\infty e\left(\frac{D}{2m}iy\right)y^{k+s-5/2}dy$$
$$=\frac{A_{\overline{\chi}}}{\mathcal{G}_{\overline{\chi}}}\frac{1}{\sqrt{m}}\left(\frac{\pi}{m}\right)^{-s-k+3/2}\Gamma\left(s+k-\frac{3}{2}\right)L\left(f_{\overline{\chi}},g;s+k-\frac{3}{2}\right)$$

provide that $s\in\mathbb{C}$ satisfies $2Re(s)>3-2k$. □

For convenience we introduce the completed Dirichlet series

$$\widetilde{\Lambda}(f_\chi,g;s)=\frac{1}{\sqrt{m}}\left(\frac{\pi}{m}\right)^{-s}\left(\frac{\pi}{N^2}\right)^{-s+k-\frac{3}{2}}\Gamma(s)\Gamma\left(s-k+\frac{3}{2}\right)$$
$$\times L(\chi^2,2s-2k+3)L(f_\chi,g;s).$$

From Lemmas 3 and (31) one has

$$\int_{\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})} h_{f_\chi,g}(\tau,z)\mathcal{E}_{N^2,\chi^2}\left(\tau,s-k+\frac{3}{2}\right)y^{-2}dqdxdpdy=\widetilde{\Lambda}(f_\chi,g;s) \tag{36}$$

for all s in $\mathbb{C}$ with $Re(s)>2k-1/2$.

Theorem 2 *Let $f(\tau,z)$ and $g(\tau,z)$ be Jacobi cusp forms of weight k and index m over the Jacobi group Γ^J.*

Let N be a positive integer and χ a Dirichlet character mod N such that both χ and χ^2 are non-principal and primitive.

Then the series $\widetilde{\Lambda}(f_\chi,g;s)$ has a holomorphic continuation to the whole complex plane and satisfies

$$\widetilde{\Lambda}(f_\chi,g;s)=\left(\frac{\mathcal{G}_\chi}{\sqrt{N}}\right)^4\widetilde{\Lambda}(f_{\overline{\chi}},g;2k-s-2).$$

Proof Let Ω be any compact subset in the s-complex plane.

By Proposition 1, the exponential decay of $h_{f_\chi,g}(\tau,z)$ as $y\to\infty$ and the description of the fundamental domain of $\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})$ given in (30), it is easy to see that $h_{f_\chi,g}(\tau,z)\mathcal{E}_{N^2,\chi^2}(\tau,s)$ is; (i) bounded on $\left(\mathcal{H}\times\mathbb{C}/\Gamma_0(N^2)\ltimes(N\mathbb{Z}\times\mathbb{Z})\right)\times\Omega$, (ii) a continuous function on (τ,z) for each s, and (iii) a holomorphic function on s for each (τ,z). These conditions imply that the integral in (36) defines a holomorphic function on the interior of Ω.

This fact implies the analytic continuation of $\widetilde{\Lambda}(f_\chi,g,s)$ to $\mathbb{C}$.

Next we prove the functional equation. From Lemma 3, the identity in Proposition 1 and Lemma 5 we have

$$\widetilde{\Lambda}(f_\chi,g;s)=\frac{A_{\overline{\chi}}\mathcal{G}_{\chi^2}}{\mathcal{G}_{\overline{\chi}}N}\widetilde{\Lambda}(f_{\overline{\chi}},g;2k-s-2).$$

Finally we recall the identity

$$\frac{A_{\overline{\chi}}\mathcal{G}_{\chi^2}}{\mathcal{G}_{\overline{\chi}}N} = \left(\frac{\mathcal{G}_\chi}{\sqrt{N}}\right)^4$$

which is proved as in [11, p. 116]. □

For our purposes we also need the analogue of Theorem 2 when χ is the principal character χ_0 (and so $N = 1$).

Theorem 3 *Let $f(\tau, z)$ and $g(\tau, z)$ be Jacobi cusp forms of weight k and index m over the Jacobi group Γ^J. Let χ_0 be the principal Dirichlet character.*

Then the series $\Gamma(s-k+3/2)^{-1}\widetilde{\Lambda}(f_{\chi_0}, g; s)$ has a holomorphic continuation to $\mathbb{C}-\{k-1/2\}$ with at most a simple pole at $s = k - 1/2$. The residue at such a point is $2^{-1} < f, g >$ where $< f, g >$ denotes the Petersson inner product of Jacobi cusp forms. Furthermore,

$$\widetilde{\Lambda}(f_{\chi_0}, g; s) = \widetilde{\Lambda}(f_{\chi_0}, g; 2k - s - 2).$$

Remark This theorem is equivalent to Corollary 1 in [7]. Notice however a counting error in the latter. The residue of $\Gamma(s - k + 3/2)^{-1}\widetilde{\Lambda}(f_{\chi_0}, g; s)$ at $s = k - 1/2$ is given without the factor 2^{-1}. The discrepancy is explained by a missing 2 in Eq. (20) of [7].

5 A twisted convolution of Siegel cusp forms

Let k be a positive integer and $F(Z) : \mathcal{H}_2 \to \mathbb{C}$ a Siegel cusp form of weight k over the group $Sp_2(\mathbb{Z})$. Then

$$F(Z) = \sum_{T \in \mathcal{J}} c(T)e(TZ) = \sum_{T \in \mathcal{J}} c(T)e(n\tau_1 + rz + m\tau_2), \tag{37}$$

where $\mathcal{J}$ is the set of half-integral, positive-definite matrices in $\mathbb{R}^{2,2}$ and the generic matrices T in $\mathcal{J}$ and Z in $\mathcal{H}_2$ are denoted as

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix}.$$

If we write the coefficients in (37) as $c(T) = c(n, r, m)$, we may consider for positive integers m, n the subseries

$$f_m(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn > r^2}} c(n, r, m)e(n\tau + rz),$$

$$\widetilde{f}_n(\tau, z) = \sum_{\substack{m,r \in \mathbb{Z} \\ 4mn > r^2}} c(n, r, m)e(m\tau + rz).$$

They are Jacobi cusp forms of weight k and index m (resp. n) over Γ^J. Clearly

$$F(Z) = \sum_{m=1}^{\infty} f_m(\tau_1, z)e(m\tau_2) = \sum_{n=1}^{\infty} \widetilde{f}_n(\tau_2, z)e(n\tau_1). \tag{38}$$

Consider the matrix

$$\mathcal{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sp_2(\mathbb{Z}). \tag{39}$$

Then $F(Z)|_k[\mathcal{I}] = F(Z)$. This implies

$$(-1)^k \sum_{n=1}^{\infty} \widetilde{f}_n(\tau_1, z) e(n\tau_2) = \sum_{m=1}^{\infty} f_m(\tau_1, z) e(m\tau_2),$$

which in turn yields $(-1)^k \widetilde{f}_m(\tau, z) = f_m(\tau, z)$ for all m.

Definition 5 Let χ (resp. ψ) be a Dirichlet character modulo N (resp. M). We define the twisted Siegel cusp form $F_{\chi,\psi}(Z)$ as

$$F_{\chi,\psi}(Z) = \sum_{\substack{m,n,r \in \mathbb{Z} \\ n>0,\, 4mn>r^2}} \chi(n)\psi(m) c(n, r, m) e(n\tau_1 + rz + m\tau_2).$$

A formal manipulation of this series yields

$$F_{\chi,\psi}(Z) = \sum_{m=1}^{\infty} \psi(m) f_{m,\chi}(\tau_1, z) e(m\tau_2) = \sum_{n=1}^{\infty} \chi(n) \widetilde{f}_{n,\psi}(\tau_2, z) e(n\tau_1). \tag{40}$$

Consequently, $F_{\chi,\psi}(\mathcal{I}Z) = (-1)^k F_{\psi,\chi}(Z)$.

Definition 6 Let $\Delta(N, M)$ be the group of matrices $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $Sp_2(\mathbb{Z})$ which satisfy the set of congruences

$$c_1 \equiv 0\,(N^2), \quad c_4 \equiv 0\,(M^2), \quad c_2 \equiv c_3 \equiv 0\,(MN), \;\; d_2 \equiv 0\,(N) \;\text{ and }\; d_3 \equiv 0\,(M).$$

These identities imply $a_3 \equiv 0$, $a_1 d_1 \equiv 1 \pmod N$ and $a_2 \equiv 0$, $a_4 d_4 \equiv 1 \pmod M$ (see [12]). In particular $\chi(\gamma) = \chi(d_1)$ and $\psi(\gamma) = \psi(d_4)$ are well-defined multiplicative characters of $\Delta(N, M)$. For convenience these characters are evaluated on the lower right entry of the group elements. Thus $\chi(\gamma)\psi(\gamma) = \overline{\chi}\psi(d_4) = \overline{\chi}\psi(D)$. Consider the matrices

$$W_N^M = \begin{pmatrix} 0 & 0 & 1/N & 0 \\ 0 & 0 & 0 & 1/M \\ -N & 0 & 0 & 0 \\ 0 & -M & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_{\nu/M}^{\mu/N} = \begin{pmatrix} 1 & 0 & \mu/N & 0 \\ 0 & 1 & 0 & \nu/M \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where μ, ν are integral parameters. They are in $Sp_2(\mathbb{Q})$.

Lemma 6 i) $\Delta(N, M) = Sp_2(\mathbb{Z}) \cap \left(W_N^M\right) Sp_2(\mathbb{Z}) \left(W_N^M\right)^{-1}$.

ii) $F_{\chi,\psi}(Z) = \frac{1}{MN} \sum_{j,\mu\,(N)} \sum_{l,\nu\,(M)} \chi(j) e\left(\frac{-j\mu}{N}\right) \psi(l) e\left(\frac{-l\nu}{M}\right) F(Z)\,|_k \left[T_{\nu/M}^{\mu/N}\right]$.

iii) *If γ is in $\Delta(N, M)$ then $F_{\chi,\psi}(Z)\,|_k[\gamma] = \overline{\chi}^2\psi^2(\gamma) F_{\chi,\psi}(Z)$. Indeed the function $F_{\chi,\psi}(Z)$ is a Siegel cusp form of weight k and character $\overline{\chi}^2\psi^2$ over $\Delta(N, M)$.*

iv) *If χ and ψ are primitive non-principal characters then*

$$F_{\chi,\psi}(Z)\,|_k \left[W_N^M\right] = (MN)^{-1}\mathcal{G}_\chi^2\mathcal{G}_\psi^2 F_{\overline{\chi},\overline{\psi}}(Z).$$

The proof of this lemma is straightforward (see [12] for similar results).

An easy consequence is that $F_{\chi,\psi}(Z)$ is then a Siegel cusp form of weight k and character $\overline{\chi}^2\psi^2$ over the group $\Delta(N, M)$.

Definition 7 Let L be a positive integer and

$$\mathcal{C}_\infty(L) = \left\{ \begin{pmatrix} {}^tA & {}^tAB \\ 0 & A^{-1} \end{pmatrix} \in Sp_2(\mathbb{Z}) \mid A \in \Gamma^0_\infty(L) \right\}$$

This is a subgroup of $Sp_2(\mathbb{Z})$. If a Siegel cusp form $H(Z) = \sum_{T\in\mathcal{J}} a(T)e(TZ)$ over a congruence subgroup of $Sp_2(\mathbb{Z})$ is invariant under $\mathcal{C}_\infty(L)$, then $a(T) = a(T')$ whenever the matrices $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ and $T' = \begin{pmatrix} n' & r'/2 \\ r'/2 & m' \end{pmatrix}$ satisfy

$$m = m', \qquad r \equiv r' \pmod{2mL} \quad \text{and} \quad \det T = \det T'.$$

Definition 8 Let $H_1(Z)$ and $H_2(Z)$ be two Siegel cusp forms of weight k over a congruence subgroup of $Sp_2(\mathbb{Z})$, invariants under $\mathcal{C}_\infty(L)$, with Fourier series $H_j(Z) = \sum_{T\in\mathcal{J}} a_j(T)e(TZ)$. Define the multiple Dirichlet series

$$L(H_1, H_2; s, w) = \sum_{\substack{T\in\mathcal{J} \\ r \pmod{2mL}}} a_1(T)\overline{a_2(T)} m^{-s}(4\det T)^{-w}. \tag{41}$$

The estimates $a_j(T) = O(T^{k/2})$ for $j = 1, 2$ hold for such Siegel cusp forms. Thus the series $L(H_1, H_2; s, w)$ is absolute and locally uniform convergent on the region $Re(s) > 2$ and $Re(w) > k + 1$.

The rest of this article is about a particular case of this convolution, e.g., the series (1) associated to the Siegel cusp forms $F(Z)$ and $G(Z)$ whose Fourier series are (37) and $G(Z) = \sum_{T\in\mathcal{J}} d(T)e(TZ)$ resp., plus the characters χ, ψ mod N and M. We use that both $F_{\chi,\psi}(Z)$ and $G(Z)$ are invariant under $\mathcal{C}_\infty(N)$.

The Fourier Jacobi series of $F_{\chi,\psi}(Z)$ is given in (40). Similarly we can write $G(Z) = \sum_{m=1}^\infty g_m(\tau_1, z)e(m\tau_2)$ where $g_m(\tau, z) = \sum_{n,r} d(n, r, m)e(n\tau + rz)$. A formal manipulation in the region of convergence yields

$$L(F_{\chi,\psi}, G; s, w) = \sum_{m=1}^\infty \psi(m)L(f_{m,\chi}, g_m; w)m^{-s}. \tag{42}$$

This equation relates the twisted convolution of Siegel cusp forms with the twisted convolution of Jacobi forms.

Our next goal is to find an integral representation for $L(F_{\chi,\psi}, G; s, w)$. For this reason we consider the Eisenstein series (12) with $\phi = \overline{\chi}^2\psi^2$. From its functional equation and the remark below Lemma 6 one has that $F_{\chi,\psi}(Z)\overline{G(Z)}E_{\overline{\chi}^2\psi^2}(Im(Z); s, w)$ is invariant under any transformation of $\mathcal{C}(N, M)$ on Z. On the other hand the volume element $(\det Y)^{-3}dXdY$ of $\mathcal{H}_2$, where $X = Re(Z)$, $Y = Im(Z)$, is invariant under $Sp_2(\mathbb{R})$. Thus the integral in the proposition below is well-defined.

Proposition 3 *Let $F(Z)$, $G(Z)$ (resp. χ, ψ) be Siegel cusp forms (resp. Dirichlet characters) as above.*

If s, w are in $\mathbb{C}$ with $Re(s) > 2$, $2Re(w) > 4k-1$ and $2Re(s+w) > 1$ then

$$\int\limits_{\mathcal{H}_2/\mathcal{C}(N,M)} F_{\chi,\psi}(Z)\overline{G(Z)}E_{\overline{\chi}^2\psi^2}\left(Y; s, -s-w-\frac{3}{2}\right)(\det Y)^{-3}dXdY$$
$$= (4\pi)^{-s-w+\frac{1}{2}}\pi^{-w}\Gamma(w)\Gamma\left(s+w-\frac{1}{2}\right)L\left(F_{\chi,\psi}, G; s, w\right). \tag{43}$$

Proof The left hand side of (43) is equal to

$$\int\limits_{\mathcal{H}_2/\mathcal{C}_\infty(N)} F_{\chi,\psi}(Z)\overline{G(Z)}p_{-s,-(-s-w-3/2)}(Y)(\det Y)^{-3}dXdY. \tag{44}$$

Consider next $\mathcal{C}_\infty = \left\{\begin{pmatrix} {}^tA & {}^tAB \\ 0 & A^{-1}\end{pmatrix} \in Sp_2(\mathbb{Z}) \mid A \in \Gamma_\infty\right\}$. This group has the coset decomposition $\mathcal{C}_\infty = \bigcup_{a\,(N)} \gamma_a \mathcal{C}_\infty(N)$ where $\gamma_a = \begin{pmatrix} {}^tA & 0 \\ 0 & A^{-1}\end{pmatrix}$ with $A = \begin{pmatrix} 1 & a \\ 0 & 1\end{pmatrix}$. Let $\mathcal{F}$ be the $\mathcal{C}_\infty$-fundamental domain in $\mathcal{H}_2$ given in [10, p. 548]. Then

$$\left\{\begin{pmatrix} \tau_1 & z \\ z & \tau_2\end{pmatrix} \mid (\tau_1, z) \in \mathcal{H} \times \mathbb{C}/\Gamma_\infty \ltimes (N\mathbb{Z}\times\mathbb{Z}),\ x_2 \in [0,1],\ y_2 \in (p^2y_1, \infty)\right\},$$

where x_j, y_j, p, q are the real coordinates determined by $\tau_1 = x_1 + iy_1$, $\tau_2 = x_2 + iy_2$ and $z = p\tau_1 + q$, is equal to the $\mathcal{C}_\infty(N)$-fundamental domain $\cup_{a\,(N)}\gamma_a^{-1}\mathcal{F}$. Using the latter we can write (44) as

$$\int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_\infty\ltimes(N\mathbb{Z}\times\mathbb{Z})}\ \int\limits_{y_2=p^2y_1}^{\infty}\ \int\limits_{x_2=0}^{1} F_{\chi,\psi}(Z)\overline{G(Z)}p_{-s,s+w}(Y)(y_2-p^2y_1)^{-\frac{3}{2}}$$
$$\times y_1^{-\frac{1}{2}}dx_2dy_2dqdx_1dpdy_1.$$

In order to compute this integral we first observe that

$$\int\limits_{x_2=0}^{1} F_{\chi,\psi}(Z)\overline{G(Z)}dx_2 = \sum_{m=1}^{\infty}\psi(m)f_{m,\chi}(\tau_1,z)\overline{g_m(\tau_1,z)}e(2miy_2).$$

Secondly, we consider the substitution $t = y_2 - p^2y_1$ and the hypothesis $2Re(s+w) > 1$ to obtain

$$\int\limits_{y_2=p^2y_1}^{\infty} e(2miy_2)(y_2-p^2y_1)^{s+w-\frac{3}{2}}dy_2 = e(2imp^2y_1)(4\pi m)^{-s-w+\frac{1}{2}}\Gamma\left(s+w-\frac{1}{2}\right).$$

These computations yield that (44) is equal to

$$(4\pi)^{-s-w+\frac{1}{2}}\Gamma\left(s+w-\frac{1}{2}\right)\sum_{m=1}^{\infty}\psi(m)m^{-s-w+\frac{1}{2}}$$
$$\times \int\limits_{\mathcal{H}\times\mathbb{C}/\Gamma_\infty\ltimes(N\mathbb{Z}\times\mathbb{Z})} e^m(2p^2iy_1)f_{m,\chi}(\tau_1,z)\overline{g_m(\tau_1,z)}y_1^{w+\frac{3}{2}-2}dqdx_1dpdy_1.$$

Finally, we use Lemma 3 and hypothesis $2Re(w) > 4k-1$ to compute the last integral. $\square$

For future references observe that the interchange of characters χ, ψ in Definition 5 gives the Siegel cusp form $F_{\psi,\chi}(Z)$ of weight k over the group $\Delta(M, N)$. An argument analogous to the one above shows that

$$\int\limits_{\mathcal{H}_2/\mathcal{C}_\infty(M)} F_{\psi,\chi}(Z)\overline{G(Z)}p_{-s,-(-s-w-\frac{3}{2})}(Y)\,(\det Y)^{-3}dXdY$$
$$= (4\pi)^{-s-w+\frac{1}{2}}\pi^{-w}\Gamma(w)\Gamma\left(s+w-\frac{1}{2}\right)L\left(F_{\psi,\chi}, G; s, w\right). \tag{45}$$

So far we have that both sides of (43) are absolute convergent and identical on the region given in Proposition 3. Our next result establishes the holomorphic continuation of $L(F_{\chi,\psi}, G; s, w)$ to a larger region.

Proposition 4 *Let $F(Z)$, $G(Z)$, χ and ψ be as in Proposition 3.*

Then $L(F_{\chi,\psi}, G; s, w)$ admits a holomorphic continuation to the region of $\mathbb{C}^2$ determined by the inequalities $Re(s) > 1$, $Re(w) > -\frac{1}{2}$.

In particular, the completed Dirichlet series $\Lambda(F_{\chi,\psi}, G; s, w)$ introduced in (2) admits a holomorphic continuation to the same region and the series $\widetilde{\Lambda}(F_{\chi,\psi}, G; s, w)$ introduced in (3) admits a holomorphic continuation to to the region $Re(s+w) > 1/2$, $Re(w) > -1/2$ with $w \neq 0$.

Proof Let $\mathcal{R}$ be the $GL_2(\mathbb{Z})$-fundamental domain in $\mathcal{P}$ known as Minkowski's reduced domain. Since $\Gamma_0^0(M, N)$ has finite index in the group $SL_2(\mathbb{Z})$, there is a positive integer ν and matrices A_j such that $GL_2(\mathbb{Z}) = \cup_{j=1}^{\nu} A_j\Gamma_0^0(M, N)$. Then $\cup_j \mathcal{R}[A_j]$ is a $\Gamma_0^0(M, N)$-fundamental domain in $\mathcal{P}$. Consider

$$\mathcal{C} = \left\{ \begin{pmatrix} {}^tA & {}^tAB \\ 0 & A^{-1} \end{pmatrix} \in Sp_2(\mathbb{Z}) \mid A \in GL_2(\mathbb{Z}) \right\}.$$

Clearly, there exists a fundamental domain V' for the action of $\mathcal{C}$ on $\mathcal{H}_2$ such that $Y \in \mathcal{R}$ whenever $Z \in V'$. Moreover, $\mathcal{C} = \cup_{j=1}^{\nu}\mathcal{C}(N, M)\gamma_{A_j}$, where $\gamma_{A_j} = \begin{pmatrix} {}^tA_j & 0 \\ 0 & A_j^{-1} \end{pmatrix}$. Hence, the integral in the left hand side of (43) is equal to

$$\sum_{j=1}^{\nu} \int\limits_{\mathcal{H}_2/\mathcal{C}} F_{\chi,\psi}(\gamma_{A_j}Z)\overline{G(\gamma_{A_j}Z)}E_{\overline{\chi}^2\psi^2}\left(Y[A_j]; s, -s-w-\frac{3}{2}\right)(\det Y)^{-3}dXdY \tag{46}$$

for $Re(s) > 1$. Next we prove that the left hand side of (43) is absolute and locally uniform convergent using the integrals in (46). Let $Re(s) = r$, $Re(w) = t$ and pick j in $\{1, 2, \ldots, \nu\}$. If $r > 1$ then

$$\left|E_{\overline{\chi}^2\psi^2}\left(Y[A_j]; s, w\right)\right| \leq \sum_{A\in SL_2(\mathbb{Z})/\Gamma_\infty} p_{-r,-t}(Y[A]).$$

As shown in [13, p. 143], there exists a real constant C (which depends locally on s, w but is independent of $Y \in \mathcal{R}$) such that the last series is bounded by $Cp_{-r,-t}(Y)$. Thus

$$\int_{\mathcal{H}_2/\mathcal{C}} \left| F_{\chi,\psi}(\gamma_{A_j} Z)\overline{G(\gamma_{A_j} Z)} E_{\overline{\chi}^2\psi^2}\left(Y[A_j]; s, w\right) (\det Y)^{-3} \right| dXdY$$
$$\leq C \int_{\mathcal{H}_2/\mathcal{C}} \left| F_{\chi,\psi}(\gamma_{A_j} Z)\overline{G(\gamma_{A_j} Z)} \right| p_{-r,-t}(Y)(\det Y)^{-3} dXdY. \tag{47}$$

Let us write $||H(Z)|| = \sum_T |a(T)e(TZ)|$ for any $H(Z) = \sum_T a(T)e(TZ)$ and $\Gamma^2(M^2N^2)$ for the principal congruence subgroup of level M^2N^2 in $Sp_2(\mathbb{Z})$. Choose also matrices $\gamma_1, \gamma_2, \ldots, \gamma_\mu$ such that $Sp_2(\mathbb{Z}) = \cup_{j=1}^{\mu} \Gamma^2(M^2N^2)\gamma_j$.

Since $F_{\chi,\psi}(Z)$ is a Siegel cusp form of weight k and trivial character on $\Gamma^2(M^2N^2)$, the function $\widetilde{F}(Z) = \sum_{j=1}^{\mu} F_{\chi,\psi}(Z)|_k[\gamma_j]$ is a Siegel cusp form of weight k and trivial character on $Sp_2(\mathbb{Z})$. In particular, there are positive real numbers c_1, c_2 such that $||\widetilde{F}(Z)|| \leq c_1 e^{-c_2 tr(Y)}$ for all Z in $\mathcal{H}_2$ with Y in $\mathcal{R}$ (see a proof of this fact in [9, p. 57]). Hence

$$|F_{\chi,\psi}(Z)| \leq ||F_{\chi,\psi}(Z)|| \leq ||\widetilde{F}(Z)|| \leq c_1 e^{-c_2 tr(Y)}$$

for all those Z. Similarly, for any matrix γ_{A_j} the map $F_{\chi,\psi}(\gamma_{A_j} Z) = \pm F_{\chi,\psi}(Z)|_k[\gamma_{A_j}]$ is a Siegel cusp form of weight k and trivial character over $\gamma_{A_j}^{-1}\Gamma^2(M^2N^2)\gamma_{A_j} = \Gamma^2(M^2N^2)$. Thus the same argument yields the inequality $|F_{\chi,\psi}(\gamma_{A_j} Z)| \leq c_1 e^{-c_2 tr(Y)}$ for all Z in $\mathcal{H}_2$ with Y in $\mathcal{R}$. From these facts and the corresponding inequality for $|G(\gamma_{A_j} Z)|$, we conclude the existence of positive real numbers c_1, c_2 such that

$$|F_{\chi,\psi}(\gamma_{A_j} Z)\overline{G(\gamma_{A_j} Z)}| \leq c_1 e^{-c_2 tr(Y)} \tag{48}$$

for all $j = 1, 2, \ldots, \nu$ and all Z in $\mathcal{H}_2$ with Y in $\mathcal{R}$. Now we use (46), (47) and (48) to get the existence of positive real numbers C and c_2 such that

$$\int_{\mathcal{H}_2/\mathcal{C}(N,M)} \left| F_{\chi,\psi}(Z)\overline{G(Z)} E_{\overline{\chi}^2\psi^2}\left(Y; s, -s-w-\frac{3}{2}\right) (\det Y)^{-3} \right| dXdY$$
$$\leq C \int_{\mathcal{H}_2/\mathcal{C}} e^{-c_2 tr(Y)} p_{-r,r+t}(Y)(\det Y)^{-\frac{3}{2}} dXdY$$
$$= C \int_{\mathcal{R}} e^{-c_2 tr(Y)} p_{-r,r+t}(Y)(\det Y)^{-\frac{3}{2}} dY. \tag{49}$$

Finally we recall the existence of a real constant C' such that $\det Y \leq y_1 y_2 \leq C' \det Y$ for all $Y = \begin{pmatrix} y_1 & v \\ v & y_2 \end{pmatrix}$ in $\mathcal{R}$. This fact and (49) imply that the absolute convergence of the integral in the left hand side of (43) follows from the convergence of

$$\int_{\mathcal{R}} e^{-By_1 - By_2} y_1^{-r} (y_1 y_2)^{r+t-\frac{3}{2}} dY \leq \int_0^\infty e^{-By_1} y_1^{t-\frac{3}{2}} dy_1 \int_{-y_1/2}^{-y_1/2} dv \int_0^\infty e^{-By_2} y_2^{r+t-\frac{3}{2}} dy_2$$

where B is some positive real number. The latter converges if $r > 1$ and $t > -\frac{1}{2}$.

The convergence of (43) that we just proved and the holomorphicity of $E_\phi(Y; s, w)$ on the region $Re(s) > 1$ give the holomorphic continuation of the right hand side of (43). From this we get the holomorphic continuation of $L(F_{\chi,\psi}, G; s, w)$ to the given region. □

6 A functional equation for $L(F_{\chi,\psi}, G; s, w)$

Consider the function $\widetilde{\Lambda}\left(F_{\chi,\psi}, G; s, w\right)$ given in (3). From Eq. (42) one has

$$\widetilde{\Lambda}\left(F_{\chi,\psi}, G; s, w\right) = \sum_{m=1}^{\infty} \psi(m)\widetilde{\Lambda}(f_{m,\chi}, g_m; w)m^{-s}$$

on the region of convergence of $L\left(F_{\chi,\psi}, G; s - w + 1/2, w\right)$. A direct application of Theorem 2 yields a functional equation.

Proposition 5 *Let $F(Z)$ and $G(Z)$ be Siegel cusp forms of weight k over $Sp_2(\mathbb{Z})$. Let M and N be relatively prime positive integers. Let χ (resp. ψ) be a Dirichlet character modulo N (resp M) such that both χ and χ^2 are primitive and non-principal. Then*

$$\widetilde{\Lambda}\left(F_{\chi,\psi}, G; s, w\right) = \left(\frac{\mathcal{G}_\chi}{\sqrt{N}}\right)^4 \widetilde{\Lambda}\left(F_{\overline{\chi},\psi}, G; s, 2k - w - 2\right). \tag{50}$$

Next we give a functional equation for $\Lambda\left(F_{\chi,\psi}, G; s, w\right)$.

Proposition 6 *Let $F(Z)$ and $G(Z)$ be Siegel cusp forms of weight k over $Sp_2(\mathbb{Z})$. Let M and N be relatively prime positive integers. Let χ (resp. ψ) be a Dirichlet character modulo N (resp. M) such that $\overline{\chi}^2\psi^2$ is primitive and non-principal. Then*

$$\Lambda\left(F_{\chi,\psi}, G; s, w\right) = (-1)^k \frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{M} \Lambda\left(F_{\psi,\chi}, G; 1 - s, s + w - \frac{1}{2}\right). \tag{51}$$

Proof Notice first that the combination of Propositions 2 and 3 yield

$$\begin{aligned} &M^s(MN)^{-(k-s-w-3/2)}(4\pi)^{-s-w+1/2}\pi^{-s-w}\Gamma(s)\Gamma(w)\Gamma\left(s + w - \frac{1}{2}\right) \\ &\times L(\overline{\chi}^2\psi^2, 2s)L\left(F_{\chi,\psi}, G; s, w\right) = \frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}} \int_{\mathcal{H}_2/\mathcal{C}(N,M)} F_{\chi,\psi}(Z)\overline{G(Z)} \\ &\times \mathcal{E}_{\chi^2\overline{\psi}^2}\left(Y^{-1}\left[\begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{M}\end{pmatrix}\right]; 1 - s, s + w - k + 1\right)(\det Y)^{k-3}dXdY. \end{aligned} \tag{52}$$

Now we focus on the Eisenstein series occurring in this integral. Since $(\det Y)Y^{-1} = Y\left[\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\right]$, we have

$$\begin{aligned} &E_\phi\left(Y^{-1}\left[\begin{pmatrix} 1/N & 0 \\ 0 & 1/M\end{pmatrix}\right]; s, w\right) \\ &= (\det Y)^{s+2w} \sum_{A \in \Gamma_0^0(M,N)/\Gamma_\infty^0(N)} \phi(a_1)p_{-s,-w}\left(Y\left[\begin{pmatrix} 0 & -1/M \\ 1/N & 0\end{pmatrix}A\right]\right) \\ &= (\det Y)^{s+2w} \sum_{B \in \Gamma_0^0(M,N)/\Gamma_0^\infty(M)} \phi(b_4)p_{-s,-w}\left(Y\left[B\begin{pmatrix} 0 & -\frac{1}{M} \\ \frac{1}{N} & 0\end{pmatrix}\right]\right). \end{aligned}$$

For the last identity we used that conjugation by $\begin{pmatrix} 0 & N \\ -M & 0 \end{pmatrix}$ defines a group isomorphism from $\Gamma_0^0(M,N)$ onto $\Gamma_0^0(M,N)$ which takes $\Gamma_\infty^0(N)$ onto $\Gamma_0^\infty(M)$. Notice also that

$$p_{-s,-w}\left(Y\left[\begin{pmatrix} 0 & -\frac{1}{M} \\ \frac{1}{N} & 0 \end{pmatrix}\right]\right) = N^{2s}(MN)^{2w} p_{-s,-w}\left(Y\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\right).$$

Hence

$$\begin{aligned} &E_\phi\left(Y^{-1}\left[\begin{pmatrix} 1/N & 0 \\ 0 & 1/M \end{pmatrix}\right]; s, w\right) \\ &= N^{2s}(MN)^{2w}(\det Y)^{s+2w} \sum_{B\in\Gamma_0^0(M,N)/\Gamma_0^\infty(M)} \phi(B) p_{-s,-w}\left(Y\left[B\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\right) \\ &= N^{2s}(MN)^{2w}(\det Y)^{s+2w} \sum_{B\in\Gamma_0^0(N,M)/\Gamma_\infty^0(M)} \overline{\phi}(B) p_{-s,-w}\left(Y\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}B\right]\right). \end{aligned}$$

In particular, using (15) and the matrix $\mathcal{I}$ defined in (39) one can write (52) as

$$\begin{aligned} &\frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}} N^{1-s}(MN)^{w-k+2}\pi^{-(1-s)}\Gamma(1-s)L(\chi^2\overline{\psi}^2, 2(1-s)) \\ &\times \int_{\mathcal{H}_2/\mathcal{C}(N,M)} \sum_{B\in\Gamma_0^0(N,M)/\Gamma_\infty^0(M)} (-1)^k F_{\psi,\chi}(\mathcal{I}Z)\overline{G(\mathcal{I}Z)}\overline{\chi}^2\psi^2(B) \\ &\times p_{-(1-s),w+2}(Im\mathcal{I}Z[B])(\det Y)^{-3}dXdY. \end{aligned}$$

Let $\mathcal{F}$ be any fundamental domain of the quotient space $\mathcal{H}_2/\mathcal{C}(N,M)$. Then its image $\mathcal{I}(\mathcal{F})$ is a fundamental domain of the quotient space $\mathcal{H}_2/\mathcal{C}(M,N)$. Consequently, the last expression is equal to

$$\begin{aligned} &(-1)^k\frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}} N^{1-s}(MN)^{w-k+2}\pi^{-(1-s)}\Gamma(1-s)L(\chi^2\overline{\psi}^2, 2(1-s)) \\ &\times \int_{\mathcal{H}_2/\mathcal{C}(M,N)} \sum_{B\in\Gamma_0^0(N,M)/\Gamma_\infty^0(M)} F_{\psi,\chi}(Z)\overline{G(Z)}\overline{\chi}^2\psi^2(B)p_{-(1-s),w+2}(Y[B])(\det Y)^{-3}dXdY, \end{aligned}$$

Next put $\gamma_B = \begin{pmatrix} {}^tB & 0 \\ 0 & B^{-1} \end{pmatrix}$, use that $F_{\psi,\chi}(Z)$ is invariant under $\Delta(M,N)$ up to the character $\chi^2\overline{\psi}^2$ and that $G(Z)$ is $Sp_2(\mathbb{Z})$-invariant to write the above as

$$\begin{aligned} &(-1)^k\frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}} N^{1-s}(MN)^{w-k+2}\pi^{-(1-s)}\Gamma(1-s)L(\chi^2\overline{\psi}^2, 2(1-s)) \\ &\times \int_{\mathcal{H}_2/\mathcal{C}(M,N)} \sum_{B\in\Gamma_0^0(N,M)/\Gamma_\infty^0(M)} F_{\psi,\chi}(\gamma_B Z)\overline{G(\gamma_B Z)}p_{-(1-s),w+2}(Im\gamma_B Z)(\det Y)^{-3}dXdY \\ &= (-1)^k\frac{\mathcal{G}_{\overline{\chi}^2\psi^2}}{\sqrt{MN}} N^{1-s}(MN)^{w-k+2}\pi^{-(1-s)}\Gamma(1-s)L(\chi^2\overline{\psi}^2, 2(1-s)) \\ &\quad\times \int_{\mathcal{H}_2/\mathcal{C}_\infty(M)} F_{\psi,\chi}(Z)\overline{G(Z)}p_{-(1-s),w+2}(Y)(\det Y)^{-3}dXdY. \end{aligned} \tag{53}$$

Finally, observe that the last integral is like the one in (44) and so equal to

$$(4\pi)^{-w}\pi^{-(s+w-1/2)}\Gamma\left(s+w-\frac{1}{2}\right)\Gamma(w)L\left(F_{\psi,\chi},G;1-s,s+w-\frac{1}{2}\right). \tag{54}$$

Putting together (52), (53) and (54) one gets a functional equation for the multiple Dirichlet series in (2). □

Now we are ready to prove our main result.

Proof of Theorem 1 First we recall that $L(F_{\chi,\psi},G;s,w)$ is a holomorphic function on the region

$$D_1=\left\{(s,w)\in\mathbb{C}^2\mid Re(s)>1,\ Re(w)>-\frac{1}{2}\right\}.$$

After Propositions 5 and 6 it suffices to show that $L(F_{\chi,\psi},G;s,w)$ admits a holomorphic continuation to $\mathbb{C}^2$. To this end we observe that the functional equation (50) allows us to define $L\left(F_{\overline{\chi},\psi},G;s,w\right)$ as a holomorphic function on

$$D_2=\left\{(s,w)\in\mathbb{C}^2\mid Re(w)>-\frac{1}{2}Re(s)+k-\frac{1}{2},\ Re(w)>2k-\frac{3}{2}\right\}.$$

Similarly, the functional equation (51) allows us to extend $L\left(F_{\overline{\chi},\psi},G;s,w\right)$ to a holomorphic function on $D_3=\left\{(s,w)\in\mathbb{C}^2\mid 0>Re(s),\ Re(w)>-Re(s)\right\}$.

Next, if we apply Eq. (50) first, (51) second, and (50) again, we get a functional equation that continues $L\left(F_{\overline{\chi},\psi},G;s,w\right)$ to a holomorphic function on the region $D_4=\left\{(s,w)\in\mathbb{C}^2\mid Re(s)>1,\ 2k-1-Re(s)>Re(w),\ 0>Re(w)\right\}$ (here we use the hypothesis $k>1$). This process shows that we have a holomorphic function on the open set $D_1\cup D_2\cup D_3\cup D_4$. Since the later is connected, we can extend $L\left(F_{\overline{\chi},\psi},G;s,w\right)$ to a holomorphic function on its convex hull (see [6, p. 41]). That is $\mathbb{C}^2$. □

Notice that the two functional equations in Theorem 1 are of order two, and they do not commute. In particular we can get new identities by composing them. If we define

$$\begin{aligned}\widetilde{\widetilde{\Lambda}}(F_{\chi,\psi},G;s,w)=2^{-2s}&\left(\frac{\pi}{M}\right)^{-2s+k-3/2}\left(\frac{\pi}{MN}\right)^{-2s+2k-3}\Gamma(s)\\&\times\Gamma\left(s+w-2k+\frac{5}{2}\right)L(\chi^2\psi^2,2s+2w-4k+5)\\&\times\Gamma\left(s-w+\frac{1}{2}\right)L(\overline{\chi}^2\psi^2,2s-2w+1)\\&\times\Gamma\left(s-k+\frac{3}{2}\right)L(\psi^2,2s-2k+3)L\left(F_{\chi,\psi},G;s-w+\frac{1}{2},w\right),\end{aligned} \tag{55}$$

and apply consecutively to it the functional equations in (51), (50) and (51) again, we obtain

$$\widetilde{\widetilde{\Lambda}}\left(F_{\chi,\psi},G;s,w\right)=\frac{\mathcal{G}_{\overline{\chi}^2\psi^2}\mathcal{G}_{\psi^2\chi^2}}{MN}\left(\frac{\mathcal{G}_{\psi}}{\sqrt{M}}\right)^4\widetilde{\widetilde{\Lambda}}\left(F_{\chi,\overline{\psi}},G;2k-s-2,w\right). \tag{56}$$

7 A non-vanishing consequence

In this last section as an application of Theorem 1, we generalize a non-vanishing result in [2]. Consider the multiple Dirichlet series (1) with χ equal to the principal character χ_0. By Theorem 3 the completed Dirichlet series $\widetilde{\Lambda}(F_{\chi_0,\psi}, G; s, w)$ defined in (3) has a meromorphic continuation to $\mathbb{C}^2$ with a simple pole at $w = k - 1/2$ for every s with $Re(s) >> 0$. In such a case,

$$Res_{w=k-1/2}\widetilde{\Lambda}(F_{\chi_0,\psi}, G; s, w) = 2^{-1}\sum_{m=1}^{\infty}\psi(m) < f_m, g_m > m^{-s}. \tag{57}$$

This is the series studied in [11,12]. If we put

$$\mathcal{D}(F, G, \psi; s, w) = \pi^{-2w+k-3/2}\Gamma(w)\Gamma\left(w-k+\frac{3}{2}\right)\zeta(2w-2k+3)\widetilde{\widetilde{\Lambda}}\left(F_{\chi_0,\psi}, G; s, w\right),$$

we get a generalization of (57). Indeed,

$$\begin{aligned}\mathcal{D}(F, G, \psi; s, w) = 2^{-2s}\left(\frac{\pi}{M}\right)^{-4s+3k-9/2}&\Gamma(s)\Gamma\left(s+w-2k+\frac{5}{2}\right)\\ &\times L(\psi^2, 2s+2w-4k+5)\Gamma\left(s-w+\frac{1}{2}\right)L(\psi^2, 2s-2w+1)\\ &\times\Gamma\left(s-k+\frac{3}{2}\right)L(\psi^2, 2s-2k+3)\sum_{m=1}^{\infty}\psi(m)\widetilde{\Lambda}(f_m, g_m; w)m^{-s}.\end{aligned}$$

Furthermore, from equation (56) one gets

$$\mathcal{D}(F, G, \psi; s, w) = \left(\frac{\mathcal{G}_{\psi}}{\sqrt{M}}\right)^4\left(\frac{\mathcal{G}_{\psi^2}}{\sqrt{M}}\right)^2\mathcal{D}(F, G, \overline{\psi}; 2k-s-2, w).$$

Using this identity and arguing as in the proof of Theorem 2 in [2] with the series

$$L(F_{\chi_0,\psi}, G; s, w) = \sum_{m=1}^{\infty}\psi(m)L(f_m, g_m; w)m^{-s}$$

instead of (57) for fixed w with $Re(w) >> 0$ one obtains Corollary 1 in the introduction, which is a direct generalization of Theorem 2 in [2].

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